Lecture 5

General Initial Surface — The Cauchy Problem

We use the Schwartz notation for a general qth order linear differential equation for a function $u : D_{\subset \mathbb{R}^n} \mapsto \mathbb{R}^n$

$$Lu = \sum_{|\alpha| \leq q} A_\alpha(x) \partial^\alpha u = B(x) \tag{1}$$

or put in form

$$Lu = \sum_{|\alpha| = q} A_\alpha(x) \partial^\alpha u + g(x, \{\partial^\alpha u\}_{|\alpha| < q}) \tag{2}$$

We write (2) explicitly

$$Lu(x_1, x_2, .., x_n) = \underbrace{\sum_{k=0}^{\alpha_1} \sum_{k=0}^{\alpha_2} ... \sum_{k=0}^{\alpha_n}}_{\substack{\alpha_1+\alpha_2+...\alpha_n \leq q \\ \alpha_n < q}} A_{(\alpha_1, \alpha_2, .., \alpha_n)}(x) \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} .. \partial_{x_n}^{\alpha_n} u(x) + g(x, \{\partial^\alpha u\}_{\sum_i^n \alpha_i \leq q-1})$$

Definition 1 (Cauchy Problem)**.** *The Cauchy problem consists of finding a solution $u(x)$ for* (1) *or* (2) *in which the Cauchy data (general initial condition) is defined on a hyper-surface $S \subset \mathbb{R}^n$ given by*

$$\phi(x_1, x_2, ..., x_n) = 0 \tag{3}$$

where $\phi \in C^q$ and the surface should be regular in the sense that

$$\bigtriangledown \phi \neq 0$$

Definition 2. *(Cauchy Data) The Cauchy data on S for a qth order equation consists of the derivatives of u of orders less than or equal to $q - 1$*

The definition above essentially means that we are to evaluate the derivatives of u with respect to x_n on S under the derivative order constraint indicated above. It must be understood that such data cannot be arbitrarily chosen, as they must satisfy certain compatibility conditions, for all functions regular near S. We will explain the phrase "computability conditions" through the following example below. We are in aim to find a solution u near S which has these Cauchy data on S. We choose $x_n = t$, which is usual said to be the 'time' derivative.

Consider

$$u_{tt} - u_t - u_x = 0, \qquad S = \{x \in \mathbb{R}^n \ : \ t = 0\} \tag{4}$$

The hyper-surface S is a special case in which it defines subspace $\mathbb{R} \times \{t = 0\} \subset \mathbb{R}^2$ where all t components are zero. In particular $S \ : \ \phi(t) = 0$ satisfying

$$\bigtriangledown \phi = (0, 1) \neq 0_{\mathbb{R}^2}$$

The derivatives of u required by Definition 2 which are to be evaluated on S (that is $t = 0$) for order $k = 0, 1$ $(\leq q - 1)$

$$u(x, 0) = \psi_0(x), \qquad u_t(x, 0) = \psi_1(x) \tag{5}$$

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$$u_x(x,0) = \varphi(x) = \underbrace{\partial_x \psi_0(x)}_{condition} \tag{6}$$

where ψ_k, φ are functions that only depend on x which are prescribed on S. Notice above that the first and third equations must be consistent with each other; this is what we mean by compatibility condition.

Definition 3. *We call S noncharacteristic is we can get all $\partial^\alpha u$ for $|\alpha| = q$ on S from a linear algebraic system consisting of the computability conditions and the differential equation* (1) *or* (2) *taken on S. We call S characteristic if it is not noncharacteristic.*

We aim to find an algebraic criterion for characteristic surfaces. We will be referring to the example above with its defined surface S. In general the Cauchy data consists of $\partial^\beta u$ with $|\beta| \leq q-1$ evaluated on S. In our case $u(x,0),\ u_t(x,0)\ u_x(x,0)$. We call the derivates with respect to x_n (in our case t) of orders less than or equal to $q-1$ the 'normal' derivatives on S. That is

$$\partial_n^k u = \partial_t^k u = \psi_k(x_1, x_2, ..., x_n) \qquad for \qquad k = 0, ..., q-1 \qquad x \in S \tag{7}$$

In our example that would be equations in (5). Meanwhile the rest of derivatives we have on S

$$\partial^\beta u = \partial_1^{\beta_1} \partial_2^{\beta_2} ... \partial_{n-1}^{\beta_{n-1}} \psi_{\beta_n} \qquad \beta_n \leq q-1 \tag{8}$$

which is in our case equation (6). We notice for $|\beta| \leq q-1$ we have all the compatibility conditions expressing the Cauchy data in terms of the normal derivatives (7).
Consider now index

$$\alpha^* = (0, ..., 0, q)$$

We see (using our example) u_{tt} cannot be expressed by (6), since the derivatives vanish. This shows that it is essential to have the PDE (1) for it is then used to express $u_{tt}(x,0)$ in terms of (8) (i.e the Cauchy data). Note that if we had other 2nd order derivatives of u they would be included in (8). We come to our conclusion that for us to uniquely determine u near S, we require $A_{\alpha^*} \neq 0$; in our example$\alpha^* = (0,2)$ so $A_{(0,2)} u_{tt}$ is the part of (1) where we have $A_{\alpha^*} = 1 \neq 0$. This is the case of (1) linear, for if it is quasi-linear, we would need to know ψ_k as it multiplies with $\partial_n^q u$.
We saw how the nature of S imposes conditions on coefficients of derivatives of u satisfying $|\alpha| = q$. The Characteristic form is defined

$$C(\xi) = \sum_{|\alpha|=q} A_\alpha \xi^\alpha \qquad \xi \in \mathbb{R}^n \tag{9}$$

We require

$$C(\bigtriangledown \phi) \neq 0 \tag{10}$$

We rewrite (2) by combining the leading derivative variable x_n to include it in the sum

$$\sum_{|\alpha|=q} A_\alpha(x) \partial^\alpha u + g(...) = 0, \qquad with \qquad A_\beta(0) \neq 0,\ \beta = (0, 0..., 0, q). \tag{11}$$

What this means is that we want this PDE to be of order q, so we want conditions $A_\beta(0) \neq 0$

Now consider a general analytic surface: A mapping operator Φ that is invertible in a nbhd of s and is analytic:

$$\Phi : \mathbb{R}^n \mapsto \mathbb{R}^n$$

$$\Phi(x) = (\phi_1(x), \phi_i(x), .., \phi_n(x)) \qquad y_i = \phi_i(x) \tag{12}$$

$D \subset \mathbb{R}^n$ we define the pre-image of D by $S = \Phi^{-1}(D) = \{x \in \mathbb{R}^n : \ \phi_n(x) = 0\}$ in nbhd of 0. We derive partial derivative expressions by chain rule

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}, \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial y_k \partial y_l}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j} + \frac{\partial u}{\partial y_k}\frac{\partial^2 y_k}{\partial x_i \partial j} \tag{13}$$

$$\sum_{|\alpha|=q} A_\alpha \partial_x^\alpha u = \sum_{|\alpha|=q} B_\alpha \partial_y^\alpha u + \textit{lower terms}. \tag{14}$$

In the specific case of $\alpha = \beta$

$$B_\beta = \sum_{|\alpha|=q} A_\alpha \left(\frac{\partial \phi_n}{\partial x_i}\right)^{\alpha_1} ... \left(\frac{\partial \phi_n}{\partial x_n}\right)^{\alpha_n}. \tag{15}$$

We want $B_\beta(0) \neq 0$.

$$C(x,\xi) = \sum_{|\alpha|=q} A_\alpha(x)\xi^\alpha \qquad (\xi \in R^n). \tag{16}$$

Characteristic form of (11)

Definition 4. *If the surface $S = \{x : \phi_n(x) = 0\} : \bigtriangledown\phi(x) \neq 0$, satisfies $C(x, \bigtriangledown\phi(x)) = 0$, then S is called a characteristic at x.*

$$\{\xi : \ C(x,\xi) = 0\} \ - \qquad \textit{Characteristic cone at } x$$

Cauchy problem for (11) has a unique analytic solution near s, if S is nowhere characteristic.
Eg. Laplace :

$$C(x,\xi) = \sum_{i=1}^{n} \xi_i^2$$

No characteristic surface.
Wave:

$$C(x,\xi) = \xi_n^2 - \sum_{i=1}^{n-1} \xi_i^2.$$

Characteristic surface is a cone.
Heat:

$$C(x,\xi) = \sum_{i=1}^{n-1} \xi_i^2$$

Characteristic cone $= \{x : \ x_1 = ...x_{n-1} = 0\}$ so Characteristic surface $= \{x : \ x_n = const\}$

Transport

$$\sum_i \alpha_i(x)\partial_i u = f, \qquad C(x,\xi) = \sum_i \alpha_i(x)\xi_i \qquad C(x, \bigtriangledown\phi(x)) = 0 \tag{17}$$

—- Suppose S is characteristic

$$\sum_{|\alpha|=q,\alpha_n<q} B_\alpha \partial_y^\alpha u + lot = 0, \qquad \text{initial data u }, \partial_m u, ...\partial_m^{q-1}u$$

Constraint in initial data.

$C-K$ theorem is local.
$C-K$ works on analytic setting.
Analytic data $\implies$ unique analytic solution.
Analytic data $\implies$ non-analytic sol. unique for linear equations.
Non-analytic data : No general theorey.

Initial data $\psi \mapsto u = S(\psi)$ solution. ψ is continuous $\implies \ \forall\epsilon > 0,\ \exists\psi_\epsilon \in C^\omega$,

$$\|\psi - \psi_\epsilon\| \le \epsilon, \qquad u_\epsilon = S(\psi_\epsilon).$$

Is there u s.t $u_\epsilon \to u$ as $\epsilon \to 0$?

Does $\psi \approx \phi$ implies $S(\psi) \approx S(\phi)$?
In general, no.

Eg. $\partial_t^2 u + \partial_x^2 u = 0 \qquad u(x,0) = 0,\ u_t(x,0) = \frac{sin(nx)}{n}$:

Proof. Assume solution form

$$u(x,t) = k(t)sin(nx).$$

$k'' - n^2k = 0 \implies k(t) = A\exp nt + B\exp^{-nt}$

$$u(x,t) = \frac{sin(nx)}{n^2}(e^{nt} - e^{-nt}) \tag{18}$$

We see that although for small values of x (ie. values near zero) our initial conditions are small, however the solution about small values of x are not bounded due to the exponential functions of t. This problem is not well posed. □

Hyperbolic Equations : transport, Wave.
Elliptic Equations: laplace.
Parabolic : heat
Dispersive: Schnodinger.
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$$\sum_{|\alpha|=q} A_\alpha \partial^\alpha u + g(...) = 0$$

Types of first order PDE's

$$A_\alpha = A_\alpha(x) \qquad semilinear$$

$$A_\alpha = A_\alpha(x, \{\partial^\beta u\}_{|\beta|<q}) \qquad quasilinear$$

$$\sum_{|\alpha|\leq q} A_\alpha(x)\partial^\alpha u = f \qquad linear.$$

$$A_\alpha(x) = const \qquad cont.ceoff.$$